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www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 315–321

INTEGRAL EQUATIONS OF DYNAMIC PROBLEMS FOR MULTILAYERED MEDIA CONTAINING A SYSTEM OF CRACKS†

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(*Received* 15 *June* 2004)

A new method of determining the dynamic characteristics of multilayered semi-bounded media with defects of the inclusion or crack type at the layer interfaces [1] is used to solve antiplane problems. Systems of integral equations of the corresponding boundary-value problems are constructed and the properties of their kernels are investigated. The dispersion curves of the determinants and matrix elements of these systems are analysed as functions of the number of layers and their elastic and geometric characteristics

1. GENERAL MATRIX-FUNCTIONAL RELATIONS

In the problem of the harmonic oscillations of a package of N plane-parallel linearly-deformable layers, which has physical-mechanical properties of crack- or cavity-type defects at the interfaces, formulae have been obtained that, in terms of Fourier transforms, express the amplitudes of the displacement vectors $\mathbf{W}_k$ of points of the medium and the stresses $\mathbf{T}_k$ at the layer interfaces as functions of the amplitudes of the vectors of the surface load $\mathbf{T}_0$ and displacement jumps $\mathbf{f}_m(\alpha, \beta)$ at the edges of the cracks [1, 2]

$$\mathbf{W}_k(\alpha, \beta, z_k) = \mathbf{K}_{N-k+1}(\alpha, \beta, z_k)\mathbf{T}_0(\alpha, \beta) + \sum_{m=1}^{N-1} \mathbf{R}_{km}(\alpha, \beta)\mathbf{f}_m(\alpha, \beta) \tag{1.1}$$

$$\mathbf{T}_k(\alpha, \beta) = \mathbf{L}_k(\alpha, \beta)\mathbf{T}_0(\alpha, \beta) + \sum_{m=1}^{N-1} \mathbf{L}_{km}(\alpha, \beta)\mathbf{f}_m(\alpha, \beta), \quad k = 1, 2, \dots, N \tag{1.2}$$

The subscript k corresponds to the interface of the k-th and $(k+1)$-th layers, m corresponds to defects on the boundary of the m-th and $(m+1)$-th layers, z_k is a local coordinate, which varies within the thickness of the k-th layer $(|z_k| \leq h_k)$, $\mathbf{T}_0 = F\mathbf{t}_0$, $\mathbf{T}_k = F\mathbf{t}_k$, $\mathbf{W}_k = F\mathbf{w}_k$, where F is the two-dimensional Fourier transform with respect to the variables x and y with parameters α and β, $\mathbf{t}_k = \{t_{1k}, t_{2k}, t_{3k}\}$ are stress vectors characterizing the interaction between the layers, and $\mathbf{w}_k = \{w_{1k}, w_{2k}, w_{3k}\}$ are the displacement vectors of points of the k-th layer.

The matrices $\mathbf{K}_n$, $\mathbf{L}_n$, $\mathbf{R}_{km}$, $\mathbf{L}_{km}$ have the uniform structure characteristic for Green's matrix-symbols of the appropriate boundary-value problems for media without defects [3]. Their elements depend on the oscillation frequency ω and on the geometric and mechanical properties of the layers: the thickness $2h_k$, the density ρ_k, the shear modulus μ_k, and Poisson's ratio ν_k.

†*Prikl. Mat. Mekh.* Vol. 69, No. 2, pp. 345–351, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.03.018

If mixed conditions are specified on the surface of the medium and at the layer interfaces

$$z = 0: \mathbf{w}_1 = \mathbf{w}_0(x, y), \quad (x, y) \in \Omega_0; \quad \mathbf{t}_0 = 0, \quad (x, y) \notin \Omega_0$$

$$z_k = -h_k: \mathbf{t}_k = \mathbf{t}_{kp}(x, y), \quad (x, y) \in \Omega_{kp}; \quad \Delta\mathbf{w}_k = 0, \quad (x, y) \notin \Omega_{kp}; \quad p = 1, 2, \ldots, M_k$$

then the matrix-functional relations (1.1) for $k = 1$ and $z_1 = h_1$, together with relations (1.2) for $k = 1, 2, \ldots, N-1$, lead to systems of integral equations for the contact stresses $\mathbf{t}_0(x, y)$ and jumps of the displacement vectors $\Delta\mathbf{w}_{kp}(x, y)$ at the edges of the cracks. Here Ω_0 is the region of contact of the punch with the surface of the medium $z = 0$, $\mathbf{w}_0$ are the displacements given in the region Ω_0, M_k is the number of cracks in the plane $z_k = -h_k$, Ω_{kp} are the regions occupied by the cracks, and $\mathbf{t}_{kp}$ are given stresses at the edges of the cracks.

We will derive such systems for the case of antiplane vibrations.

2. FUNCTIONAL RELATIONS DESCRIBING ANTIPLANE OSCILLATIONS

We will consider the problem of harmonic oscillations of a package of N plane-parallel ideally elastic layers of thickness $H = 2(h_1 + h_2 + \ldots + h_N)$ with rigidly restrained lower face and occupying a volume $-H \le z \le 0, -\infty \le x, y \le +\infty$ (h_k is the half-thickness of the k-th layer). At the interfaces of the physical-mechanical parameters there are defects of the crack type, situated in the regions

$$\Omega_{km} : \{z_k = -h_k, \; a_{km} \le x \le b_{km}, -\infty < y < +\infty\}, \quad m = 1, 2, \ldots, M_k, \quad k = 1, 2, \ldots, N-1$$

The surface of the medium is subject to a certain dynamical action characterized by the vector of distributed stresses $\mathbf{t}_0(x, y)e^{-i\omega t}$, which is either given or may be determined by solving a contact problem.

We shall assume that the given and unknown vector quantities have only one non-zero component, which does not depend on the y coordinate or, in terms of Fourier transforms, on the parameter β:

$$\mathbf{T}_0 = \{0, T_0(\alpha), 0\}, \; \mathbf{W}_k = \{0, W_k(\alpha, z_k), 0\}, \; \mathbf{T}_k = \{0, T_k(\alpha), 0\}, \; \mathbf{f}_k = \{0, f_k(\alpha), 0\}$$

In that case the matrix relations (1.1) and (1.2) become functional relations and the construction of the solution is simplified considerably.

In terms of Fourier transforms, we will express Green's functions of packages of m layers ($m = 1, 2, \ldots, N$) rigidly coupled with an undeformable base as ratios of entire functions:

$$G_m(z) = \frac{k_m(z)}{\Delta_m}, \quad -H_m \le z \le 0, \quad H_m = 2\sum_{n=1}^{m} h_{N-n+1}$$

Note that $k_m(z)$ and Δ_m depend on the parameter α of the Fourier transform, the frequency of harmonic oscillations ω, and the geometric and mechanical parameters of layers N, $N-1$, etc. to $N-m+1$, inclusive. Throughout, in order to abbreviate the notation, the only argument indicated in functional relations will be that guaranteeing their unambiguous interpretation.

In the case of antiplane oscillations, the matrices occurring in formulae (1.1) and (1.2) are replaced by the corresponding functions

$$K_{N-k+1}(h_k) = (-1)^{k+1}\frac{k_{N-k+1}(h_k)}{\mu_k\Delta_N}$$

$$L_k = (-1)^{k+1}\frac{\Delta_{N-k}}{\Delta_N}, \quad L_{km} = (-1)^{k+m-1}\frac{1}{\Delta_N}\begin{cases}\mu_m\Delta_{N-k}R_m(h_m), & k > m\\ \mu_k\Delta_{N-m}R_k(h_k), & k \le m\end{cases}; \quad \Delta_0 \equiv 1 \qquad (2.1)$$

$$R_{km} = \begin{cases} L_m, \quad k = 1 \\ (-1)^{k+m}(\mu_m/\mu_k)R_m(h_m)k_{N-k+1}(h_k)/\Delta_N, \quad k \ne 1, \quad k > m \\ (-1)^{k+m-1}D_{k-1}(h_{k-1})\Delta_{N-m}/\Delta_N, \quad k \ne 1, \quad k \le m \end{cases}$$

where $R_k(h_k)$ and $D_k(h_k)$ are defined by the recurrence formulae

$$R_1(h_k) = \sigma_{2k}\mathrm{sh}(2\sigma_{2k}h_k), \quad D_1(h_k) = \mathrm{ch}(2\sigma_{2k}h_k)$$

$$R_k(h_k) = R_1(h_k)D_{k-1}(h_{k-1}) + g_{k-1}D_1(h_k)R_{k-1}(h_{k-1})$$

$$D_k(h_k) = D_1(h_k)D_{k-1}(h_{k-1}) + g_{k-1}\sigma_{2k}^{-2}R_1(h_k)R_{k-1}(h_{k-1})$$

$$\sigma_{2k} = \sqrt{\alpha^2 - \frac{\rho_k}{\mu_k}\omega^2}, \quad g_{k-1} = \frac{\mu_{k-1}}{\mu_k}; \quad k = 2, 3, \ldots, N$$

If mixed conditions are prescribed at the surface of the medium and the layer interfaces, the required system of integral equations is set up from the relations

$$W_1(h_1) = K_N(h_1)T_0 + \sum_{m=1}^{N-1} L_m f_m, \quad T_k = L_k T_0 + \sum_{m=1}^{N-1} L_{km} f_m, \quad k = 1, 2, \ldots, N-1$$

$$f_m(\alpha) = \sum_{p=1}^{M_m} F(\Delta w_{mp})$$

We define a matrix $\mathbf{K}(\alpha) = \|K_{ij}\|_{i,j=1}^{N}$ with elements

$$K_{11} = K_N(h_1), \quad K_{1j} = K_{j1} = L_{j-1}, \quad K_{ij} = L_{(i-1)(j-1)}; \quad i, j = 2, 3, \ldots, N$$

and integral operators

$$\mathcal{K}(\Omega)q = \int_\Omega k(x-\xi)q(\xi)d\xi, \quad k(x) = \frac{1}{2\pi}\int_\delta K(\alpha)e^{-i\alpha x}d\alpha$$

$$\mathcal{L}_q(t_0, \Delta w_{km}) = \mathcal{K}_{q1}(\Omega_0)t_0 + \sum_{k=1}^{N-1}\sum_{m=1}^{M_k} \mathcal{K}_{q(k+1)}(\Omega_{km})\Delta w_{km}, \quad q = 1, 2, \ldots, N$$

The choice of the contour δ is dictated by the radiation principle [4]. The matrix $\mathbf{K}(\alpha)$ will be called the matrix-symbol of the system of integral equations just constructed.

In the notation we have adopted, the integral equation of dimension $M + 1$ ($M = M_1 + M_2 + \ldots + M_{N-1}$ is the total number of cracks in the medium) may be written in the form

$$\mathcal{L}_1(t_0, \Delta w_{km}) = w_0(x), \quad x \in \Omega_0; \quad \mathcal{L}_{p+1}(t_0, \Delta w_{km}) = t_{pn}(x), \quad x \in \Omega_{pn}$$
$$n = 1, 2, \ldots, M_p; \quad p = 1, 2, \ldots, N-1$$

These equations enable us to investigate various aspects of the dynamics of a multilayered base.

Setting $f_m(\alpha) = 0$ for all $m = 1, 2, \ldots, N-1$, we obtain a contact problem for a multilayered base without defects, arriving at the well-known one-dimensional integral equation

$$\mathcal{K}_{11}(\Omega_0)t_0 = w_0(x), \quad x \in \Omega_0$$

Taking $T_0(\alpha) = 0$, we obtain the dynamic problem of the oscillations of a multilayered medium generated by oscillations of only the edges of the cracks, and the corresponding system of convolution integral equations

$$\sum_{k=1}^{N-1}\sum_{m=1}^{M_k} \mathcal{K}_{(p+1)(k+1)}(\Omega_{km})\Delta w_{km} = t_{pn}(x), \quad x \in \Omega_{pn}$$
$$n = 1, 2, \ldots, M_p; \quad p = 1, 2, \ldots, N-1 \tag{2.2}$$

Since this problem is of independent interest, we re-denote the matrix symbol of the last system by $\mathbf{L}(\alpha) = \|L_{ij}\|_{i,j=1}^{N-1}$. It is obvious that $\mathbf{L}(\alpha)$ is obtained from $\mathbf{K}(\alpha)$ by eliminating the first row and the first column.

Note that integral equations (2.2), considered with the same mechanical parameters for all the layers, give the solution of the dynamic problem for a homogeneous layer with a system of cracks in the planes $z_p = -h_p$.

Having functional relations and integral equations of the problems for a package of layers, it can easily be generalized to the case of a layered half-space. When that is done, the general appearance of the notation is the same, but when the elements of the matrix-symbols $\mathbf{K}^{\infty}(\alpha)$ and $\mathbf{L}^{\infty}(\alpha)$ are defined, formulae (2.1) must be considered with

$$k_1(h_N) = 1, \quad \Delta_1(h_N) = \sigma_{2N}$$

Putting

$$k_1(h_N) = 1, \quad \Delta_1(h_N) = \sigma_{2N}, \quad D_1(h_1) = 1, \quad R_1(h_1) = \sigma_{21}$$

in these relations, we obtain functional relations and the matrix $\mathbf{L}_{\infty}^{\infty}(\alpha)$ for a layered space.

3. EXAMPLE: THE CASE $N = 3$

Forming the system of functional equations for $N = 3$, we have:

the displacements of points of the medium surface

$$W_1(h_1) = (k_3(h_1)T_0/\mu_1 - \Delta_2 f_1 + \Delta_1 f_2)/\Delta_3 \tag{3.1}$$

the stresses at the layer interfaces

$$\begin{aligned} T_1 &= (-\Delta_2 T_0 - \mu_1 R_1(h_1)\Delta_2 f_1 + \mu_1 R_1(h_1)D_1(h_3)f_2)/\Delta_3 \\ T_2 &= (\Delta_1 T_0 + \mu_1 R_1(h_1)D_1(h_3)f_1 - \mu_2 R_2(h_2)D_1(h_3)f_2)/\Delta_3 \end{aligned} \tag{3.2}$$

To construct the system of integral equations, we rewrite relations (3.1) and (3.2) in the form

$$\mathbf{W}(\alpha) = \mathbf{K}(\alpha)\mathbf{Q}(\alpha) \tag{3.3}$$

where

$$\mathbf{Q} = \{T_0, f_1, f_2\}, \quad \mathbf{W} = \{W_1(h_1), T_1, T_2\}, \quad \mathbf{K}(\alpha) = \|K_{ij}\|_{i,j=1}^{3}$$

Under these conditions,

$$\det\mathbf{K} = \mathrm{ch}(2\sigma_{21}h_1)\varphi(h_2, h_3)/\Delta_3, \quad \varphi(h_2, h_3) = \mu_2\sigma_{22}\mathrm{sh}(2\sigma_{22}h_2)\mathrm{ch}(2\sigma_{23}h_3)$$

Using Eqs (3.3), we obtain integral equations and systems of integral equations for a variety of problems. We present the system of integral equations in the general case, when

$$T_0(\alpha) \neq 0, \quad f_1(\alpha) \neq 0, \quad f_2(\alpha) \neq 0$$

We have

$$\mathcal{L}_1(t_0, \Delta w_{km}) = w_0(x), \quad x \in \Omega_0, \quad \Omega_0 : \{z = 0, |x| \leq a, -\infty < y < +\infty\}$$

$$\mathcal{L}_2(t_0, \Delta w_{km}) = t_{1n}(x), \quad a_{1n} \leq x \leq b_{1n}, \quad n = 1, 2, \ldots, M_1$$

$$\mathcal{L}_3(t_0, \Delta w_{km}) = t_{2n}(x), \quad a_{2p} \leq x \leq b_{2p}, \quad p = 1, 2, \ldots, M_2$$

Putting $f_1(\alpha) = 0$ (or $f_2(\alpha) = 0$), we obtain a system of integral equations for the case of a single crack or a system of cracks situated in a three-layered medium only in the plane $z = -2h_1$ (or only in the plane $z = -2h_1 - 2h_2$).

When $T_0(\alpha) = 0$, only relations (3.2) participate in the formation of the system of integral equations; they may be written in matrix form as

$$\mathbf{T}(\alpha) = \mathbf{L}(\alpha)\mathbf{f}(\alpha), \quad \mathbf{T} = \{T_1, T_2\}, \quad \mathbf{f} = \{f_1, f_2\}$$

where

$$\det\mathbf{L} = \mu_1\sigma_{21}\,\mathrm{sh}(2\sigma_{21}h_1)\varphi(h_2, h_3)/\Delta_3, \quad \Delta_3 = \Delta_3(\alpha, \omega, \mu_k, \rho_k, h_k), \quad k = 1, 2, 3$$

4. PROPERTIES OF THE MATRIX-SYMBOLS OF SYSTEMS OF INTEGRAL EQUATIONS

In order to determine classes of well-posedness and to construct solutions of systems of integral equations, it is necessary to study the properties of the elements of their matrix-symbols. It is most important to describe the asymptotic behaviour of the elements as $|\alpha| \to \infty$ and to investigate the behaviour of the real zeros and poles (dispersion curves) of the elements and the determinants of these matrices in the $(\mathrm{Re}\alpha, \omega)$ plane.

It has been established that the matrices $\mathbf{K}(\alpha)$, $\mathbf{L}(\alpha)$ are symmetric and may be represented in the form

$$\mathbf{K}(\alpha) = \frac{1}{\Delta_N}\|k_{ij}(\alpha)\|_{i,j=1}^{N}, \quad \mathbf{L}(\alpha) = \frac{1}{\Delta_N}\|l_{ij}(\alpha)\|_{i,j=1}^{N-1}$$

The elements $k_{ij}(\alpha)$, $l_{ij}(\alpha)$ are entire even functions of the parameter α; Δ_N is the denominator of Green's function G_N for a multilayered package without defects.

For the elements of the matrix $K(\alpha)$ on the contour δ we have the following asymptotic estimates as $|\alpha| \to \infty$

$$K_{11}(\alpha) = \frac{|\alpha|^{-1}}{\mu_1}[1 + O(|\alpha|^{-2})], \quad K_{ii}(\alpha) = -\frac{\mu_{i-1}}{1+g_{i-1}}|\alpha|[1 + O(|\alpha|^{-2})], \quad i = 2, 3, \ldots, N$$

$$K_{1j}(\alpha) = (-1)^{j+1}P_{1j}(\alpha)[1 + O(|\alpha|^{-2})], \quad j = 2, 3, \ldots, N$$

$$K_{ij}(\alpha) = (-1)^{i+j+1}\frac{\mu_i}{1+g_{i-1}}P_{ij}(\alpha)|\alpha|[1 + O(|\alpha|^{-2})], \quad i \neq j \neq 1$$

where

$$P_{ij}(\alpha) = \frac{2^{j-i}}{\prod_{k=i}^{j-1}(1+g_k)}\exp\left(-|\alpha|\sum_{k=i}^{j-1}2h_k\right)$$

The asymptotic behaviour of the element K_{11} for a multilayered medium is identical with the asymptotic behaviour of the function $G_1(\alpha, \mu_1)$, while that of the remaining diagonal elements K_{ii} is determined by the asymptotic behaviour of the function $-G_1^{-1}(\alpha, \mu_{i-1}\mu_i/(\mu_{i-1} + \mu_i))$.

The method we have used yields relations convenient for numerical analysis not only of the elements but also of the determinants of the matrices

$$\det\mathbf{K}(\alpha) = \frac{D_1(h_1)\det\mathbf{L}(\alpha)}{\mu_1 R_1(h_1)}, \quad \det\mathbf{L}(\alpha) = (-1)^{N-1}\frac{D_1(h_N)}{\Delta_N}\prod_{k=1}^{N-1}\mu_k R_1(h_k)$$

Note that in the case of a single crack in the plane $z_m = -h_m$ of an N-layered medium, we have

$$\det\mathbf{K}(\alpha) = (-1)^{N-1}\Delta_m(h_1, h_2, \ldots, h_m)\Delta_{N-m}(h_N, h_{N-1}, \ldots, h_{m+1})/\Delta_N$$

Hence it follows that if the following conditions are satisfied

$$\Delta_{N-m}(h_N, h_{N-1}, \ldots, h_{m+1}) = \Delta_n(h_1, h_2, \ldots, h_n)$$

$$\Delta_{N-n}(h_N, h_{N-1}, \ldots, h_{n+1}) = \Delta_m(h_1, h_2, \ldots, h_m) \quad (n+m=N)$$

then the determinants of the matrices corresponding to a single crack in the plane $z_n = -h_m$ or in the plane $z_n = -h_n$ have the same zeros. In a homogeneous medium, the determinants of the matrix **K** with the crack situated at the level $z = -h$ or at the level $z = -H + h$ are equal.

We also note that if a single crack is situated in the plane $z = -H/2$ of a homogeneous layer, then all the zeros of the element K_{22}, with the exception of $\alpha^2 = \rho_1\omega^2/\mu_1$, coincide with the zeros of K_{11}, that is, with the zeros of Green's function of a layer without defects.

5. NUMERICAL RESULTS

For the case considered in Section 3 – a three-layered medium – we present the results of numerical analysis of the dispersion curves of the elements (Fig. 1) and the determinants (Fig. 2) of the matrix **K**, as a function of the geometric and mechanical parameters of the problem with $\nu_i = 0.3$, $\rho_i = 1$ $(i = 1, 2, 3)$, $H = 1$, $2h_1 = h_2 = 2h_3 = 1/4$; the dimensionless values of the shear moduli are indicated in the appropriate parts of the figures; the curves of the poles are represented by the solid curves.

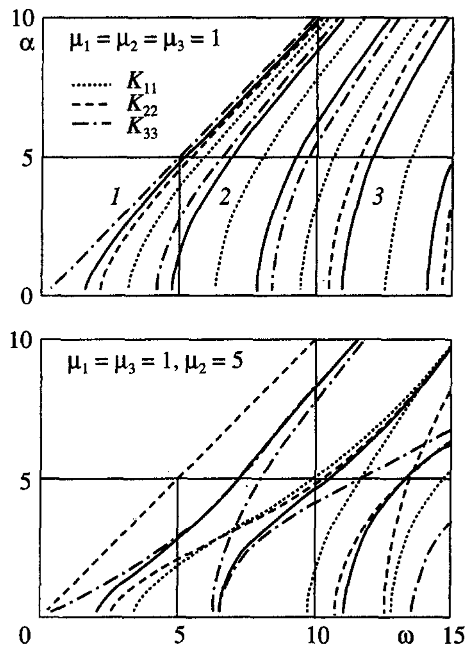

Fig. 1

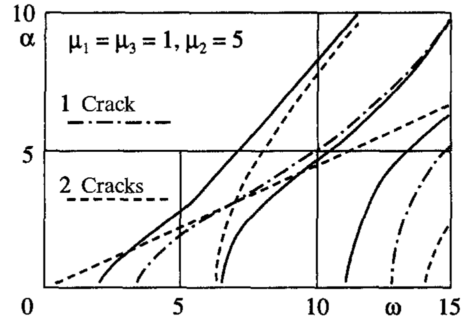

Fig. 2

For the diagonal elements of the matrices, as in the case of a defect-free medium, one observes alternation of zeros and poles (Fig. 1). When there is a system of cracks in a homogeneous layer, the diagonal elements may have identical zeros. Thus, curves *2* and *3* in Fig. 1 are common to the elements K_{11}, K_{22} and K_{33}, and curve *1* is common to the elements K_{22} and K_{33}.

In Fig. 2 we show dispersion curves of the determinants $\mathbf{K}(\alpha, \omega)$ corresponding to the presence of a single crack in the plane $z = -H/2$ and two cracks in the planes $z = -H/4$, $z = -3H/4$. In the case of one crack in a layered medium, $\det\mathbf{K}(\alpha, \omega)$ has zeros and poles beginning from some point ω^*. If there are two or more cracks in the medium, a curve of zeros occurs emanating from the origin. It is characteristic that for a crack in the middle of a homogeneous layer, all the zeros of the determinant coincide with the odd-numbered zeros of the element K_{11} (with the curves numbered as they appear on the axis $\alpha = 0$), that is, with the odd-numbered zeros of Green's function for a defect-free medium. For a layered medium, this is true only for a certain symmetry of the mechanical and geometric parameters of the problem, as for example in the case shown in Fig. 2.

We wish to thank V. A. Babeshko for proposing the topic of this paper, for advice and for discussing the results.

This research was supported financially by the Russian Foundation for Basic Research (03-01-00694, 03-01-96537 and 03-01-96645), the Russian Ministry of Education (E-02.4.0-191), the Federal Special-Purpose "Integration" programme (B0121), and the "State Support for Leading Scientific Schools" programme (NSh-2107-2003.1).

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Translated by D.L.